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Stability of Cauchy Horizons

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We prove that for any 3-dimensional compact hypersurface S in a noncompact 4-dimensional space-time manifold M, $S \subset M$, the set of Lorentzian metrics on M for which S is a partial Cauchy surface and Cauchy horizon of S is nonempty contains a nonempty open subset (in compact-open topology). This result indicates that the set of metrics admitting Cauchy horizons originating from compact hypersurfaces is large.

KEY WORDS: spacetime geometry; general relativity; Cauchy horizons.

1. INTRODUCTION

The future boundary of domain of dependence, if non-empty, is called the Cauchy horizon and it is a null surface. To the future of a Cauchy horizon space-time cannot be predicted from the initial surface S. A fundamental unresolved problem in classical relativity posed by Roger Penrose is whether there is a "cosmic censor" that ensures existence of an initial surface from which the whole of space-time is predictable and a Cauchy horizon does not occur. Hence it is of interest to study the properties of Cauchy horizons. In this paper we study stability of Cauchy horizons arising from compact Cauchy surfaces. We investigate whether there exists an open set in the space of Lorentzian metrics for which such horizons arise. Our main result is that for compact partial Cauchy surfaces an open set exists in compact-open topology.

The topological stability of Cauchy horizons has already been investigated by several authors. Beem (1995) has shown locational stability for compact subsets of the horizon when the spacetime satisfies the strong causality condition and he has proven that for a very general class of space-times there is a stability of topological type of the horizon for nearby metrics with wider light cones. Chruściel

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and Isenberg (1997) have proven stability of compact Cauchy horizons whose null generators admitted a global Poincaré section and if a certain global quantity which they call Q was sufficiently large.

2. PRELIMINARIES

Definition 1. A space-time (M, g) is a smooth 4-dimensional, connected Hausdorff manifold M with a semi-Riemannian metric g of signature $(-, +, +, +)$, a countable basis, and a time orientation.

A set S is said to be *achronal* if there are no two points of S with timelike separation.

Definition 2. The *future Cauchy development* $D^+(S)$ consists of all points $p \in M$ such that each past endless and past directed causal curve from p intersects the set S. The *future Cauchy horizon* is $H^+(S) = \overline{(D^+(S))} - I^-(D^+(S))$, where $I^-(D^+(S))$ is the chronological past of $D^+(S)$.

We give definitions and state our results in terms of the future horizon $H^+(S)$, but similar results hold for any past Cauchy horizon $H^-(S)$.

Some other basic definitions and concepts concerning causal and topological structure of space-time can be found in (Beem *et al.*, 1996; Hawking and Ellis, 1973).

3. STABILITY OF COMPACT CAUCHY HORIZONS

We shall first introduce a map between the space of Riemannian metrics and the space of Lorentzian metrics and we shall give some of its properties.

Definition 3. Let V be a 4-dimensional vector space and let $e^0 \in V$, $e^0 \neq 0$. Let $V^* \otimes_S V^*$ be a symmetrized tensor product. Let $W \subset V^* \otimes_S V^*$ denote the open set of such symmetric tensors h that $h(e^0, e^0) \neq 0$.

$\phi_{e^0} : W \to W$ is a map given by

$$\phi_{e^0}(h)(x, y) = h(x, y) - \frac{2h(e_0, x)h(e_0, y)}{h(e_0, e_0)}.$$

The above map has the following properties:

- The map ϕ_{e^0} is continuous.
- The map is involutive, i.e. $\phi^2_{e^0} = I$.
- If h is a scalar product then $\phi_{e^0}(h)$ is a Lorentzian metric.
- If h is a Lorentzian metric and $h(e_0, e_0) < 0$ then $\phi_{e^0}(h)$ is a scalar product.
- For arbitrary $\lambda \in R$, $\lambda \neq 0$, $\phi_{\lambda e^0} = \phi_{e^0}$, and $\phi_{e^0}(\lambda h) = \lambda\phi_{e^0}(h)$.

- Let h be a scalar product; then $x \in V$ is a vector such that the angle between x and e^0 is less than 45° if and only if x is timelike with respect to the Lorentzian metric $\phi_{e^0}(h)$.

Let us remark that the above definition can be naturally extended to the case when e^0 is a vector field on M and we have $T^*M \otimes_S T^*M$ instead of $V^* \otimes_S V^*$. We shall denote the extended map also by ϕ.

The properties of ϕ given above apply correspondingly.

Definition 4. Let Φ_{e^0} denote the mapping given by the formula

$$\Phi_{e^0}(\sigma) := \phi_{e^0}\sigma$$

where the domain of Φ_{e^0} is the set of sections $\sigma \in C^k(T^*M \otimes T^*M)$ such that $\sigma(e^0, e^0)|_x \neq 0$ at every $x \in M$.

It follows as a corollary of the above given properties of ϕ_{e^0} that

- Φ_{e^0} is involutive, and hence injective.
- the image of Φ_{e^0} is equal to its domain.
- if σ is a Riemannian metric, then $\Phi_{e^0}(\sigma)$ is a Lorentzian metric such that e^0 is timelike.
- conversely, if σ is a Lorentzian metric such that e^0 is timelike, $\Phi_{e^0}(\sigma)$ is a Riemannian metric.
- as a consequence, for every Lorentzian metric σ such that e^0 is timelike, there exists a Riemannian metric $\hat{\sigma}$ such that $\sigma = \Phi_{e^0}(\hat{\sigma})$.

On the function space one often introduces a compact-open topology which can be defined in the following way:

Let $C^k(M)$ denote the set of all real-valued functions of class $C^k (k \geq 0)$ on the manifold M and let $K \subset M$ be an arbitrary compact subset and $O \subset R$ be an arbitrary open subset.

The sets $\Omega_{K,O} = \{f : M \to R \mid f(K) \subset O\}$ constitute a subbasis of compact-open topology in $C^k(M)$.

For the case of sections of a bundle (like a space of metrics) a suitable generalization requires care: it should be independent of the choice of trivialization. This condition is fulfilled by the following definition.

Definition 5. Let $E(\pi, M)$ be a vector bundle where E is the bundle space, M is the base and π is the projection. Let O be an open subset of E and let K be a compact subset of M. Subbasis of the compact-open topology in the space $C^k(E, \pi, M)$ of C^k-sections ($k \geq 0$) of the bundle is generated by the sets of the form $\Omega_{K,0} = \{\sigma \in C^k(E, \pi, M) \mid \sigma(K) \subset O\}$.

Theorem 1. *Let M be a compact manifold (possibly with boundary) and let $E = T^*(M) \otimes T^*(M)$ be a symmetrized tensor product of two copies of cotangent bundles over M and e_0 be a nowhere vanishing vector field on M.*

1. *The set of Riemannian metrics on M is an open subset of the space of sections of E in the compact-open topology.*
2. *Φ_{e^0} is continuous.*
3. *The image of the space of Riemannian metrics on M under Φ_{e^0} is an open set in the space of Lorentzian metrics on M.*

Proof: Let O be the subset of E consisting of all strictly positive-definite tensors over all points of the base manifold. The set O is open in E because the set of strictly positive-definite tensors at each point is an open subset of the corresponding fiber, and the bundle is locally trivial. Property 1 follows from the compactness of the base manifold.

The continuity of Φ_{e^0} follows from the continuity of ϕ_{e^0}: consider a compact subset $K \subset M$ and an open set $U_1 \subset E$. Let Ω_{K,U_1} denote the set of sections of E that restricted to K take values in U_1. It suffices to observe that $\Phi_{e^0}^{-1}(\Omega_{K,U_1}) = \Omega_{K,\phi^{-1}(U_1)}$.

The image of the set of Riemannian metrics on M under Φ_{e^0} is the set of Lorentzian metrics such that e^0 is a timelike vector field. Let $p \in M$. The function $\pi^{-1}(p) \to R$ such that $\pi^{-1}(p) \ni g_p \mapsto g_p(e_p^0, e_p^0)$ is continuous (indeed, polynomial) and consequently the condition $g_p(e_p^0, e_p^0) < 0$ defines an open subset in $\pi^{-1}(p)$. Repeating the above construction for every fiber we obtain an open subset U in E. The sections that are elements of the image of Φ_{e^0} restricted to Riemannian metrics take values in the set U. Because the manifold M is compact we have that the image of the set of Riemannian metrics under Φ_{e^0} is open.

Proposition 1. *Let M_2 be a 4-dimensional compact manifold (possibly with boundary) and M_1 be a compact submanifold of M_2. The restriction of Riemannian metrics defined on M_2 to M_1, regarded as a mapping between spaces of sections of the appropriate tensor bundles, is continuous with respect to the compact-open topology and onto.*

Proof: For simplicity of notation we denote by (F, π, M_1) the restriction of the bundle (E, π, M_2) of symetric tensors on M_2. Let $\Omega_{K,O}$ be a certain element of the subbasis of the space $C(F, \pi, M_1)$ of continuous sections of the bundle (F, π, M_1). We shall prove that counterimage of $\Omega_{K,O}$ under the restriction map is open in the space of sections $C(E, \pi, M_2)$. Let O' be the complement of the set O in F. Since M_1 is a closed subset of M_2, F is closed in E. It follows that O' is closed in E, by virtue of being a closed subset of a closed subset of E. Consequently the complement U of O' in E is an open set. It is easy to see that the set of sections of the bundle (E, π, M_2) which on the set K take values in U is exactly equal to the counterimage of $\Omega_{K,O}$ under the restriction map. Since the constructed set of sections is open in the compact-open topology in $C(E, \pi, M_2)$ the restriction map is continuous.

The fact that every Riemannian metric defined on a compact submanifold M_1 of M_2 can be extended to all of M_2 follows from a standard construction using the partition of unity.

Proposition 2. *Let V_1 be a 4-dimensional vector space and let V_2 be a 3-dimensional vector subspace of V_1. Let $e_0 \in V_1$ and let $e_0 \notin V_2$. Then the set O of Riemannian metrics on V_1 on which the angle between e_0 and the normal to V_2 is greater than 45° is open.*

The claim follows from the continuity of the angle and the normal with respect to the metric.

Proposition 3. *Let us consider the restriction of the tangent bundle $T(S \times [0,1])$ to $S \times \{1\}$ and let $V_1(x)$ denote the fiber of this bundle at $x \in S \times \{1\}$ and $V_2(x) \subset V_1(x)$ be the space tangent to $S \times \{1\}$ at x. Then the sum $\mathcal{O}$ over x of sets $O(x)$ defined for each point $x \in S \times \{1\}$ as in Proposition 2, is an open subset of the total space of the bundle E restricted to $S \times \{1\}$.*

The statement follows from the local triviality of the bundle E and from the fact the $O(x)$ is an open subset of the fiber at each point x.

Lemma 1. *The set of Riemannian metrics defined on the set $S \times [0,1]$ such that the angle between e_0 and normal to $S \times \{1\}$ is greater than 45° at every point of $S \times \{1\}$ is open.*

At each point of $S \times \{1\}$, the cosine of the angle between two nonvanishing vectors is given by a continuous function of the coefficients of the metric at that point. Imposing an inequality on the value of this angle at a point defines an open subset of the fiber of the tensor bundle over this point. Repeating this for all points of $S \times \{1\}$ yields an open subset O_1 of the total space of the tensor bundle restricted to $S \times \{1\}$. $\Omega_{S\times\{1\},O_1}$ forms an open subset of the space of sections of the restricted bundle. The counterimage of this subset under the restriction map is open in the space of Riemannian metrics on $S \times [0,1]$ by virtue of Proposition 3, proving the claim.

Since the set $\mathcal{O}$ is open according to Proposition 3 then the set of sections of the bundle E restricted to $S \times \{1\}$ taking values in $\mathcal{O}$ is open in the compact-open topology.

We are now ready to state and prove our main result.

Theorem 2. *For any noncompact* 4*-dimensional space-time manifold* M *and any* 3*-dimensional compact hypersurface* $S \subset M$ *the set of Lorentzian metrics on* M *for which* S *is a partial Cauchy surface and Cauchy horizon of* S *is nonempty contains a nonempty open subset (in compact-open topology).*

The strategy of the proof is based on the following idea. We construct a set $\mathcal{G}$ of Riemannian metrics on $S \times [0, 1]$ and the vector field e^0 such that for every $g \in \mathcal{G}$ $\Phi_{e^0}(g)$ is a Lorentzian metric which has a closed timelike curve contained in $S \times \{1\}$. We shall prove that $\mathcal{G}$ is open. From the existence of a closed timelike curve in $S \times \{1\}$ it follows that $\Phi_{e^0}(g)$ has a Cauchy horizon in $S \times (0, 1)$.

Proof: If the future of S is not of the form $S \times R^+$ then the Cauchy horizon of S in not empty, since the domain of dependence of S is homeomorphic to $S \times R^+$. Thus it is enough to restrict our attention to the case when future of S is topologically $S \times R^+$.

Let us consider the region $S \times [0, 1]$ and let $e^0 = \partial/\partial t$ where t is the coordinate along the interval $[0, 1]$. Let us take a neighbourhood of $S \times \{1\}$ and the set of metrics for which the angle between $\partial/\partial t$ and normal to $S \times \{1\}$ is greater than 45° at every point of $S \times \{1\}$. By Lemma 1 the above inequality defines an open subset in the space of Riemaniann metrics on $S \times [0, 1]$.

Considering $S \times \{1\}$ as a closed submanifold in $S \times [0, 1]$ we have by Proposition 3 that the counterimage of $\mathcal{O}$ (where $\mathcal{O}$ is the set defined in Proposition 3) under bundle restriction is an open subset of the set of Riemannian metrics on $S \times [0, 1]$.

Using the map $\Phi_{\frac{\partial}{\partial t}} : \hat{g} \to g$ we obtain Lorentzian metrics g with the property that metric induced on $S \times \{1\}$ has the signature $(-1, 1, 1)$. It follows from compactness of S that for every such metric there exists a closed timelike curve in $S \times \{1\}$. Hence there must be a Cauchy horizon in the set $S \times (0, 1)$.

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